Problem Set 9 - Solution - LV 141.A55 QISS

1. Persistent-current qubit

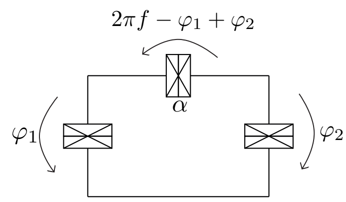

Figure 1: Persistent-current qubit

(a) Introducing

$$f = 2\pi \frac{\phi_{\text{ext}}}{\phi_0}$$

Left junction

$$I_1 = I_0 \sin(\varphi_1) + C\dot{V}_1 = I_0 \sin(\varphi_1) + C\frac{\phi_0}{2\pi}\ddot{\varphi}_1$$

Right junction

$$I_2 = I_0 \sin(\varphi_2) + C\dot{V}_2 = I_0 \sin(\varphi_2) + C\frac{\phi_0}{2\pi}\ddot{\varphi}_2$$

Top junction

$$I_3 = \alpha I_0 \sin(2\pi f + \varphi_2 - \varphi_1) + \alpha C(\dot{V}_2 - \dot{V}_1) = \alpha I_0 \sin(2\pi f + \varphi_2 - \varphi_1) + \alpha C\frac{\phi_0}{2\pi}(\ddot{\varphi}_2 - \ddot{\varphi}_1)$$

Kirchhoff equations

$$I_1 = I_3 = -I_2$$

$$\begin{aligned}
\mathcal{L} &= \mathcal{L}(\varphi_1, \varphi_2, \dot{\varphi}_1, \dot{\varphi}_2) \\
&= \frac{C}{2}\left(\frac{\phi_0}{2\pi}\right)^2 \dot{\varphi}_1^2 + \frac{C}{2}\left(\frac{\phi_0}{2\pi}\right)^2 \dot{\varphi}_2^2 + \alpha\frac{C}{2}\left(\frac{\phi_0}{2\pi}\right)^2 (\dot{\varphi}_1 - \dot{\varphi}_2)^2 \\
&\quad + \frac{\phi_0 I_0}{2\pi}\cos(\varphi_1) + \frac{\phi_0 I_0}{2\pi}\cos(\varphi_2) + \alpha\frac{\phi_0 I_0}{2\pi}\cos(\varphi_1 - \varphi_2 - 2\pi f) \\
&= E_c\left(\frac{C\phi_0}{2\pi e}\right)^2 \dot{\varphi}_1^2 + E_c\left(\frac{C\phi_0}{2\pi e}\right)^2 \dot{\varphi}_2^2 + \frac{E_c}{\alpha}\left(\frac{\alpha C\phi_0}{2\pi e}\right)^2 (\dot{\varphi}_1 - \dot{\varphi}_2)^2 \\
&\quad + E_J\cos(\varphi_1) + E_J\cos(\varphi_2) + \alpha E_J\cos(\varphi_1 - \varphi_2 - 2\pi f)
\end{aligned}$$

Verifying that the Lagrange equations reproduce the Kirchhoff equations:

$$\begin{aligned}
0 &= \frac{d}{dt}\left\{\frac{\partial \mathcal{L}}{\partial \dot{\varphi}_1}\right\} - \frac{\partial \mathcal{L}}{\partial \varphi_1} \\
&= C\left(\frac{\phi_0}{2\pi}\right)^2 \ddot{\varphi}_1 + \alpha C\left(\frac{\phi_0}{2\pi}\right)^2 (\ddot{\varphi}_1 - \ddot{\varphi}_2) \\
&\quad \frac{\phi_0 I_0}{2\pi}\sin(\varphi_1) + \alpha\frac{\phi_0 I_0}{2\pi}\sin(\varphi_1 - \varphi_2 - 2\pi f) \\
&= \frac{\phi_0}{2\pi}(I_1 - I_3)
\end{aligned}$$

$$\begin{aligned}
0 &= \frac{d}{dt}\left\{\frac{\partial \mathcal{L}}{\partial \dot{\varphi}_2}\right\} - \frac{\partial \mathcal{L}}{\partial \varphi_2} \\
&= C\left(\frac{\phi_0}{2\pi}\right)^2 \ddot{\varphi}_2 - \alpha C\left(\frac{\phi_0}{2\pi}\right)^2 (\ddot{\varphi}_1 - \ddot{\varphi}_2) \\
&\quad \frac{\phi_0 I_0}{2\pi}\sin(\varphi_2) - \alpha\frac{\phi_0 I_0}{2\pi}\sin(\varphi_1 - \varphi_2 - 2\pi f) \\
&= \frac{\phi_0}{2\pi}(I_2 + I_3)
\end{aligned}$$

(b)

$$\begin{aligned}
q_1 &= \frac{\partial \mathcal{L}}{\partial \dot{\varphi}_1} = C\left(\frac{\phi_0}{2\pi}\right)^2 \dot{\varphi}_1 + \alpha C\left(\frac{\phi_0}{2\pi}\right)^2 (\dot{\varphi}_1 - \dot{\varphi}_2) \\
q_2 &= \frac{\partial \mathcal{L}}{\partial \dot{\varphi}_2} = C\left(\frac{\phi_0}{2\pi}\right)^2 \dot{\varphi}_2 - \alpha C\left(\frac{\phi_0}{2\pi}\right)^2 (\dot{\varphi}_1 - \dot{\varphi}_2)
\end{aligned}$$

$$\begin{aligned}
\dot{\varphi}_1 &= \frac{1}{C\left(\frac{\phi_0}{2\pi}\right)^2}\left(\frac{1+\alpha}{1+2\alpha}q_1 + \frac{\alpha}{1+2\alpha}q_2\right) \\
\dot{\varphi}_2 &= \frac{1}{C\left(\frac{\phi_0}{2\pi}\right)^2}\left(\frac{\alpha}{1+2\alpha}q_1 + \frac{1+\alpha}{1+2\alpha}q_2\right)
\end{aligned}$$

(c)

$$\begin{aligned}
\mathcal{H} &= \mathcal{H}(\varphi_1, \varphi_2, q_1, q_2) \\
&= \dot{\varphi}_1 q_1 + \dot{\varphi}_2 q_2 - \mathcal{L} \\
&= 4\frac{E_c}{\left(2e\frac{\phi_0}{2\pi}\right)^2}\left(\frac{1+\alpha}{1+2\alpha}q_1^2 + \frac{2\alpha}{1+2\alpha}q_1 q_2 + \frac{1+\alpha}{1+2\alpha}q_2^2\right) \\
&\quad -E_J\cos(\varphi_1) - E_J\cos(\varphi_2) - \alpha E_J\cos(\varphi_1 - \varphi_2 - 2\pi f)
\end{aligned}$$